Kuwait University	Math 102	January 13, 2004
Dept of Math and Comp Sci	Final Exam	Duration: 2 hours

CALCULATORS, MOBILE PHONES AND PAGERS ARE NOT ALLOWED

1. Given that $f(x) = x + \ln(e + \tanh x)$.

   (a) Explain why the domain of f is $(-\infty, \infty)$. **(1 point)**

   (b) Show that f is one-to-one on its domain. **(2 point)**

   (c) Show that the point $P(1,0)$ is on the graph of f^{-1}, and find the equation of the tangent line to the graph of f^{-1} at P. **(3 points)**

2. (a) Use logarithmic differentiation to find $\left.\dfrac{dy}{dx}\right|_{x=0}$ if

$$y = \frac{|x+1|^{\sin^{-1} x}}{(e^{-x^2} + \operatorname{sech} x)\sqrt[3]{2^{-x} + x^2}}$$ **(3 points)**

   (b) Evaluate $\displaystyle\lim_{x\to\infty} \tan^{-1}(\ln x - \sinh x)$ **(3 points)**

3. Evaluate the following integrals (4 points each)

   (a) $\displaystyle\int \frac{1}{(3 - x^2 - 2x)^{\frac{3}{2}}}\,dx$ (b) $\displaystyle\int (\sin x)\ln(\sin x)\,dx$

   (c) $\displaystyle\int \frac{\sqrt{\tan x} + \sin x}{\cos^4 x}\,dx$ (d) $\displaystyle\int \frac{2x^2 - x + 8}{x\,(x^2+4)}\,dx$

4. Determine if $\displaystyle\int_1^3 \frac{1}{x^2\sqrt{x^2-1}}\,dx$ is convergent or divergent. If convergent, find its value. **(3 points)**

5. Find the area of the region that is inside the graphs of both polar equations $r = \sin\theta$ and $r = \sin 2\theta$. **(3 points)**

6. (a) Find the equation of the sphere whose one diameter has the end points $A(5,-3,-2)$ and $B(-1,1,4)$. **(3 points)**

   (b) Find the equation of the plane that contains $P(5,0,2)$ and the line $x = 4 + 3t$, $y = 3 - 2t$, $z = -3 + t$. **(3 points)**

1. $f(x) = x + \ln(e + \tanh x)$

(a) $-1 < \tanh x < 1\ \forall x$, and $e > 1 \Rightarrow e + \tanh x > 0\ \forall x$. Thus, $\ln(e + \tanh x)$ is well-defined for all x, so $D_f = (-\infty, \infty)$.

(b) $f'(x) = 1 + \frac{\operatorname{sech}^2 x}{e+\tanh x} > 0$ for all $x \Rightarrow f$ is increasing on $(-\infty, \infty)$. Thus, f is one-to-one on $(-\infty, \infty)$.

(c) $f(0) = \ln e = 1 \Rightarrow f^{-1}(1) = 0$, so $P(1, 0)$ is on the graph of f^{-1}.

$f'(0) = 1 + \frac{1}{e} = 1 + e^{-1}$

$\left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{1+e^{-1}}$

Tangent line: $\quad y = \frac{1}{1+e^{-1}}(x-1) \quad$ or $\quad (1+e^{-1})y - x + 1 = 0$

2. (a) $\ln y = (\sin^{-1} x)\ln|x+1| - \ln(e^{-x^2} + \operatorname{sech} x) - \frac{1}{3}\ln(2^{-x} + x^2)$

$\frac{y'}{y} = \frac{1}{\sqrt{1-x^2}}\ln|x+1| + \frac{\sin^{-1} x}{x+1} - \frac{-2xe^{-x^2} - (\operatorname{sech} x)(\tanh x)}{e^{-x^2} + \operatorname{sech} x} - \frac{-2^{-x}\ln 2 + 2x}{3(2^{-x}+x^2)}$

$y(0) = \frac{1}{2}$, so $y'(0) = \frac{1}{2}(-\frac{-\ln 2}{3(1)}) = \frac{\ln 2}{6}$

(b) $\ln x - \sinh x = (\ln x)\left(1 - \frac{\sinh x}{\ln x}\right)$

$\frac{\sinh x}{\ln x}$ has the form $\frac{\infty}{\infty}$ at ∞, and $\lim_{x\to\infty} \frac{(\sinh x)'}{(\ln x)'} = \lim_{x\to\infty} (x\cosh x) = \infty$. By L'Hospital's rule $\lim_{x\to\infty} \frac{\sinh x}{\ln x} = \infty$, so that $\lim_{x\to\infty} (\ln x - \sinh x) = \lim_{x\to\infty} (\ln x)(1 - \frac{\sinh x}{\ln x}) = -\infty$.

Thus, $\lim_{x\to\infty} \tan^{-1}(\ln x - \sinh x) = \tan^{-1}\left(\lim_{x\to\infty}(\ln x - \sinh x)\right) = -\frac{\pi}{2}$.

3. (a) $(3 - x^2 - 2x)^{\frac{3}{2}} = (3 + 1 - (x^2 + 2x + 1))^{\frac{3}{2}} = (4 - (x+1)^2)^{\frac{3}{2}}$

$x + 1 = 2\sin\theta \qquad dx = 2\cos\theta d\theta$

$\int \frac{1}{(4-(x+1)^2)^{\frac{3}{2}}}dx = \int \frac{2\cos\theta}{(4\cos^2\theta)^{\frac{3}{2}}}d\theta = \frac{1}{4}\int \frac{1}{\cos^2\theta}d\theta = \frac{1}{4}\int \sec^2\theta d\theta = \frac{1}{4}\tan\theta + C = \frac{x+1}{4\sqrt{3-x^2-2x}} + C$

(b) $u = \ln(\sin x) \qquad dv = \sin x dx \qquad du = \frac{\cos x}{\sin x}dx \qquad v = -\cos x$

$$\begin{aligned}\int (\ln(\sin x))\sin x dx &= uv - \int v du = -(\ln(\sin x))(\cos x) + \int \frac{\cos x}{\sin x}\cos x dx\\ &= -(\ln(\sin x))(\cos x) + \int \frac{1-\sin^2 x}{\sin x}dx = -(\ln(\sin x))(\cos x) + \int(\csc x - \sin x\\ &= -(\ln(\sin x))(\cos x) + \ln|\csc x - \cot x| + \cos x + C\end{aligned}$$

(c) $\int \frac{\sqrt{\tan x}+\sin x}{\cos^4 x}dx = \int \frac{\sqrt{\tan x}}{\cos^4 x}dx + \int \frac{\sin x}{\cos^4 x}dx$

$$\begin{aligned}\int \frac{\sqrt{\tan x}}{\cos^4 x}dx &= \int \sqrt{\tan x}\sec^2 x d(\tan x) = \int \sqrt{\tan x}(1 + \tan^2 x)d(\tan x) \overset{u=\tan x}{=} \int u^{1/2}(1+u^2)du\\ &= \tfrac{2}{3}u^{3/2} + \tfrac{2}{7}u^{7/2} + C = \tfrac{2}{3}(\tan x)^{3/2} + \tfrac{2}{7}(\tan x)^{7/2} + C_1\end{aligned}$$

$\int \frac{\sin x}{\cos^4 x}dx = \int (\cos x)^{-4}\sin x dx = -\int (\cos x)^{-4} d(\cos x) = \frac{1}{3}(\cos x)^{-3} + C_2 = \frac{1}{3}\sec^3 x + C_2$

(d) $\frac{2x^2-x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow (A+B)x^2 + Cx + 4A = 2x^2 - x + 8$.

Thus, $A = 2, \quad B = 0, \quad C = -1$.

$\int \frac{2x^2-x+8}{x(x^2+4)}dx = \int \frac{2}{x}dx - \int \frac{1}{x^2+4}dx = 2\ln|x| - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

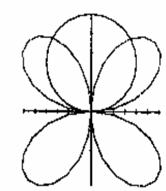
4. $x = \sec u \qquad dx = \sec u \tan u du$

$\int \frac{1}{x^2\sqrt{x^2-1}}dx = \int \frac{1}{\sec^2 u \tan u}\sec u \tan u du = \int \cos u du = \sin u = \frac{\sqrt{x^2-1}}{x}$

$\int_1^3 \frac{1}{x^2\sqrt{x^2-1}}dx = \lim_{c\to 1}\int_c^3 \frac{1}{x^2\sqrt{x^2-1}}dx = \lim_{c\to 1}(\frac{2\sqrt{2}}{3} - \frac{\sqrt{c^2-1}}{c}) = \frac{2\sqrt{2}}{3}$

5. The curves are drawn below. They intersect in the first quadrant when $\theta = 0, \frac{\pi}{3}$. By symmetry

$$\begin{aligned}\text{Area} &= 2\times(A_1 + A_2) = \int_0^{\pi/3}\sin^2\theta d\theta + \int_{\pi/3}^{\pi/2}\sin^2 2\theta d\theta = \int_0^{\pi/3}\tfrac{1}{2}(1-\cos 2\theta)d\theta + \int_{\pi/3}^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)d\theta\\ &= \tfrac{1}{2}\left(\tfrac{\pi}{2} - \left[\tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} - \left[\tfrac{1}{4}\sin 4\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right) = \tfrac{1}{2}(\tfrac{\pi}{2} - \tfrac{1}{2}\sin\tfrac{2\pi}{3} + \tfrac{1}{4}\sin\tfrac{4\pi}{3}) = \tfrac{1}{2}(\tfrac{\pi}{2} - \tfrac{1}{2}\tfrac{\sqrt{3}}{2} - \tfrac{1}{4}\tfrac{\sqrt{3}}{2}) = \tfrac{\pi}{4} - \tfrac{3\sqrt{3}}{16}\end{aligned}$$



6. (a) Center is $C(2, -1, 1)$. Radius $= \frac{1}{2}d(A, B) = d(C, A) = d(C, B) = \sqrt{22}$

Equation is: $\quad (x-2)^2 + (y+1)^2 + (z-1)^2 = 22 \quad$ or $\quad x^2 + y^2 + z^2 - 4x + 2y - 2z - 16 = 0$

(b) The plane also contains $Q(4, 3, -3)$. The vector $a = \langle 1, -3, 5\rangle$ corresponds to $\overrightarrow{QP}$. The vector $b = \langle 3, -2, 1\rangle$ is in the direction of the line.

A normal to the plane $= a \times b = \begin{vmatrix} i & j & k \\ 1 & -3 & 5 \\ 3 & -2 & 1 \end{vmatrix} = 7i + 14j + 7k$

Equation is: $\quad 7(x-5) + 14(y-0) + 7(z-2) = 0 \quad$ or $\quad x + 2y + z - 7 = 0$

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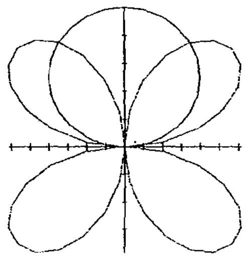

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